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The Game of Set and How it Fits Into Mathematics

Introduction

Most undergraduate mathematics students are probably not familiar with Set. They are likely familiar with the concept of a set in the context of Set Theory, but in this paper, a reference to Set is not a reference to Set Theory. Set is a card game invented by Marsha Jean Falco, a population geneticist, in 1974 while studying epilepsy in German Shepherds.[1] At its core, Set is a game about pattern recognition, but the rules of the game can reveal surprisingly intricate mathematics. Set has close connections with topics such as cap set analysis which is on the cutting edge of mathematics, as well as other concepts such as combinatorics. Computer simulation is also commonly used when analyzing the mathematics of Set, as well as mathematics in general. These connections allow mathematicians to see complex problems in a new perspective and facilitate generalizations that can be applied in a myriad of subjects.

Methods

Numerous articles focused on the mathematical analysis of the card game Set were gathered through means such as IUPUI's University Library OneSearch option, Google, and Google Scholar. Searching for mathematical articles about the card game Set had some challenges because of a couple reasons. Firstly, the word "set" is very common in English. Also, there is a branch of mathematics called "Set Theory" that serves as an integral part of the language that mathematicians use to communicate mathematical ideas, but is not directly related to the game Set. Luckily, the game Set is relatively popular among mathematicians. Phrases used in the searches include "card game set," "math card game set," and "math analysis set game." All of these phrases returned thousands of results unrelated to the actual card game Set, but relevant articles were found.

To determine if an article contained relevant information, it was simply confirmed that the article was indeed focused on the mathematics of the card game Set. With this strategy, 30 initial sources were found, and the list of sources was narrowed down to 10 articles that focused on either cap set analysis, combinatorics, or computer simulation. The 10 articles were then analyzed in a grid based on the aforementioned topics. This made it possible to effectively see general trends between each topic.

Results

Before any research can be analyzed, it is necessary to understand the rules of Set.

Set Rules

Set is a card game with a custom 81-card deck where each card has 4 attributes: number, shading, color, and shape with 3 possibilities for each attribute as shown in Table 1.

Number: { One, Two, Three }

Shading: { Solid, Striped, Open }

Color: { Red, Green, Purple }

Shape: { Ovals, Squiggles, Diamonds }

Table 1.[1]

The game can be played on your own or with others, and gameplay consists of 12 cards being dealt face up in front of everyone and the goal is to find 3 cards that form a "set." Three cards make up a set if, with respect to each of the four attributes, the cards are either all the same or all different.[1] For example, these three cards make a set because every attribute is all different:

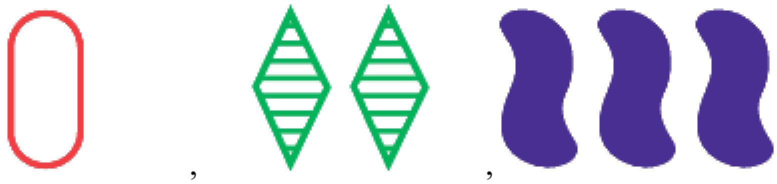

Figure 1.[2]

But these cards *don't* make a set because the color is not all different or all the same:

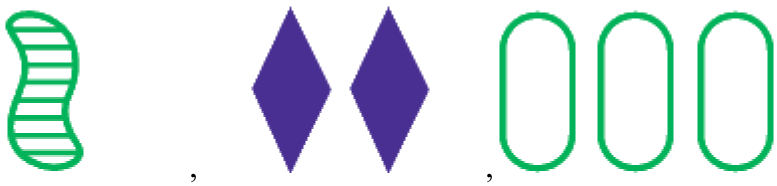

Figure 2.[2]

As a final example, these cards *do* make a set because the number is all different and every other attribute is all the same:

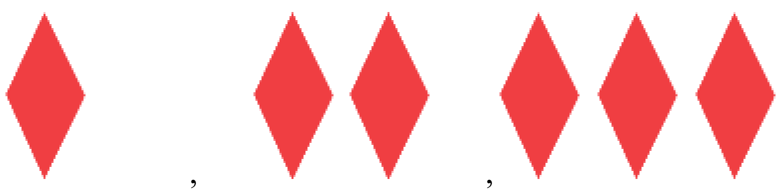

Figure 3.[2]

When someone finds a set, a point is awarded to that person and the 3 cards are removed and replaced with new cards from the deck. It is possible for no set to exist in the 12 dealt cards. In this case, more cards are dealt.

Cap Set Analysis

In Set, a card can be completely described as a point in the space of 4-tuple integers modulo 3. That is to say: a 4-dimensional space where each axis only has the values 0, 1, or 2. This is because each card has 4 qualities and 3 possibilities per quality. For example, a card with 3 red, solid, squiggles could be described as the point (2, 0, 0, 1) where 2 corresponds with 3, 0 with red, 0 with solid, and 1 with squiggles. This aligns with the mappings shown previously in Table 1.

The numbers that we assign each property are arbitrary, so different systems could have the same card correspond with a different point, but the idea is that we can make a one-to-one map between cards and points. With this model, the property of 3 cards forming a set reduces to the property of 3 points lying on the same line. This model also allows us to translate the question of “how many cards must be laid out to guarantee there is at least one set?” to “how many 4-dimensional points must be picked to guarantee there is at least one line through 3 points?”[1] This model also brings us to cap set analysis. A cap set is a set of points in the space of n-tuple integers modulo 3 such that no 3 points lie on the same line. So, the question of how many cards are necessary to guarantee a set can be answered by asking “how big can a 4-dimensional cap set be?” because we are in 4 dimensions. This was calculated to be 20, which means that if there are no sets in a group of cards, there are at most 20 cards in that group.[1] This implies that in a group of 21 cards, it is guaranteed that there is at least one set.

Cap set analysis is not only useful in answering questions about the game Set, but it is also part of the cutting edge of mathematics. Terence Tao, a Fields Medal winner who is “widely regarded as one of the greatest living mathematicians,” is quoted as saying: “Perhaps my favourite open question is the problem on the maximal size of a cap set.” in a blog post that he wrote about the topic.[3,4]

The maximal size of a cap set in 3 dimensions was proven to be 9 by Raj Chandra Bose in 1947, 20 in 4 dimensions by Giuseppe Pellegrino in 1971, 45 in 5 dimensions by Yves Edel in 2002, and 112 in 6 dimensions by Aaron Potechin. Dimensions 7 and higher are not yet known.[5]

Combinatorics

In the context of Set, combinatorics can be used in a wide variety of ways. For example, the calculations of maximal cap set sizes can be done using combinatorics.[1] Combinatorics is also used in statistical analysis of Set. This analysis can be as simple as proving that given any 2 cards, there is exactly one card that forms a set with all 3 cards.[6] Most of the non-trivial statistical analysis of Set involves calculating the probability that certain events happen. For example, the odds of there being no set in 12 cards is 30:1 on the first round, and then sharply declines to around 13:1 and 14:1 for the remaining rounds.[7] Another example is the fact that around 31% of all Set games will always have a set among the 12 cards.[7] While these are interesting facts, they don't particularly aid in the furthering of the mathematical theory of Set or Set-adjacent concepts.

In the context of more mathematically pure analysis of Set, combinatorics is generally used as a proof method. A hand can be defined somewhat intuitively as a collection of distinct cards, and a useful isomorphism can be defined between hands that makes it easy to combinatorially count the number of distinct (non-isomorphic) hands of any size. The aforementioned isomorphism involves creating a bijection between hands as well as between the attributes of the cards in each hand to determine if two hands are isomorphic.[6] This specific strategy yields that there is 1 "type" of 1-card hand, 4 types of 2-card hands, and 20 types of 3-card hands.[6] The pattern described with this isomorphism turns out to be the "number of ternary codes of length 4 with *n* words" as documented in the OEIS (On-Line Encyclopedia of Integer Sequences) as sequence A034216.[6,8]

Computer Simulation

Computer simulations can be used to help calculate previously mentioned combinatorial probabilities. This is how the odds of certain events happening were calculated as shown in the previous section. Henrik Warne wrote a program in both Ruby and Java that went through all of the possibilities during play to come up with the aforementioned numbers as mentioned in his blog post.[7] Decorated mathematician and computer scientist Donald Knuth also wrote a computer simulation that finds all nonisomorphic collections of Set cards that contain no sets.[9] Generally, computers are often used in mathematics to easily confirm or deny intuition about a concept, but more recently have been used to aid in proofs.

Computer simulations are also used to automate the gameplay of Set. There are various versions of the game that can be found online, but there are also programs that specialize in finding a set in a collection of cards.[2] This calculation is known to be NP-complete.[10]

Programs have also been developed that allow detection of cards from Set via an image taken from a camera.[11] This process includes taking in an image as input and detecting the edges of the cards in the image so that they can be analyzed individually to determine what card they are. This is done by using machine learning classifiers such as number of shapes, color, and color.[11] This process is done with a neural network approach as opposed to a deep learning approach that could be more robust, as the former can easily fail given the supplied image has too much noise. [11]

Discussion

This paper highlights how mathematical analysis can be applied to the game of Set and how that application leads to useful mathematical theory. As the discussion of various studies has shown, the game Set has deep connections with cutting edge parts of mathematics such as cap set analysis. The generalization of Set to *n* attributes yields an exact link to the analysis of *n*-tuple spaces modulo 3. Analysis of Set through the lens of combinatorics as well as computer simulation also produces great insight into their respective fields such as computer vision and game simulation. Further research into the game Set could allow mathematicians to gain insight into greater problems such as the maximal size of a cap set. Although Set is a finite game with simple rules, the hidden mechanics that lie behind it are incredibly valuable to mathematics as a whole.

References

1. Davis L, Maclagan D. 2003. The card game set. *The Mathematical Intelligencer* 25:33–40.
2. Jensen G. 2024. Set. *Greg Jensen*. https://gregjensen.dev/projects/set.html
3. Terence Tao. *Wikipedia*. https://en.wikipedia.org/wiki/Terence_Tao
4. Tao T. 2007. Open question: best bounds for cap sets. *What's New*. https://terrytao.wordpress.com/2007/02/23/open-question-best-bounds-for-cap-sets/
5. Bolognese C. 2018. The Mathematical Beauty of the Game SET. *The Aperiodical*. https://aperiodical.com/2018/07/the-mathematical-beauty-of-the-game-set/
6. Schneider J. 2020. Toward a Combinatorial Theory of SET and Related Card Games. *Arxiv:2010.01882*.
7. Warne H. 2011. SET® Probabilities Revisited. *Henrik Warne's blog*. https://henrikwarne.com/2011/09/30/set-probabilities-revisited/
8. The On-Line Encyclopedia of Integer Sequences® (OEIS®). 2024. *A034216*. https://oeis.org/A034216
9. Knuth D. 2001. SETSET. *Knuth: Programs*. https://www-cs-faculty.stanford.edu/~knuth/programs.html
10. Set (card game). *Wikipedia*. https://en.wikipedia.org/wiki/Set_(card_game)
11. Ditrih H, Grgić S, Turković L. 2021. Real-Time Detection and Recognition of Cards in the Game of Set. *International Symposium ELMAR*.